

UPPER TRIANGULAR TOEPLITZ MATRICES AND REAL PARTS OF QUASINILPOTENT OPERATORS

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ABSTRACT. We show that every self-adjoint matrix B of trace 0 can be realized as $B = T + T^*$ for a nilpotent matrix T with $\|T\| \leq K\|B\|$, for a constant K that is independent of matrix size. More particularly, if D is a diagonal, self-adjoint $n \times n$ matrix of trace 0, then there is a unitary matrix $V = XU_n$, where X is an $n \times n$ permutation matrix and U_n is the $n \times n$ Fourier matrix, such that the upper triangular part, T , of the conjugate V^*DV of D satisfies $\|T\| \leq K\|D\|$. This matrix T is a strictly upper triangular Toeplitz matrix such that $T + T^* = V^*DV$. We apply this and related results to give partial answers to questions about real parts of quasinilpotent elements in finite von Neumann algebras.

1. INTRODUCTION

It is well known and easy to show (by induction) that every self-adjoint matrix whose trace vanishes is unitarily equivalent to a matrix having zero diagonal; therefore, it is equal to the real part of a nilpotent operator.

Recall that an element z of a Banach algebra is quasinilpotent if its spectrum is $\{0\}$, and that this is equivalent to $\lim_{n \rightarrow \infty} \|z^n\|^{1/n} = 0$. Fillmore, Fong and Sourour showed [4] that a self-adjoint operator T on an infinite dimensional separable Hilbert space can be realized as the real part $(Z + Z^*)/2$ of a quasinilpotent operator Z if and only if 0 is in the convex hull of the essential spectrum of T .

Since each quasinilpotent element of a II_1 -factor has trace equal to zero (by, for example, Proposition 4 of [7]) the following question seems natural:

Question 1.1. If \mathcal{M} is a II_1 -factor with trace τ and if $a = a^* \in \mathcal{M}$ has $\tau(a) = 0$, must there be a quasinilpotent operator $z \in \mathcal{M}$ with $a = z + z^*$?

Analogously, the following question is also natural:

Question 1.2. If \mathcal{M} is a finite type I von Neumann algebra, and if $a = a^* \in \mathcal{M}$ has center-valued trace equal to zero, must there be a quasinilpotent operator $z \in \mathcal{M}$ with $a = z + z^*$?

An answer to Question 1.2 will, necessarily, and an answer to Question 1.1 will, most likely, involve a quantitative understanding of the problem in matrix algebras. The main result of this paper (Theorem 2.7) is a step in this direction.

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Our interest in quasinilpotent operators in II_1 -factors is partially motivated by the paper [5] of Haagerup and Schultz. In it, they show that every element of a II_1 -factor whose Brown measure is not concentrated at a single point, has a nontrivial hyperinvariant subspace. Since the support of the Brown measure is contained in the spectrum, quasinilpotent operators are examples of those to which the Haagerup-Schultz theorem does not apply and, indeed, the hyperinvariant subspace problem remains open for quasinilpotent operators in II_1 -factors.

The following result is a straightforward consequence of Theorem 8.1 of [5].

Theorem 1.3 ([5]). *For any element T of a finite von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$,*

$$A := \text{s.o.t.-}\lim_{n \rightarrow \infty} ((T^*)^n T^n)^{1/2n} \quad (1)$$

exists, and $\text{supp}(\mu_T) = \{0\}$ if and only if $A = 0$.

The notation in (1) is for the limit in strong operator topology on $\mathcal{B}(\mathcal{H})$. This result characterizes those operators to which the Haagerup-Schultz result on existence of hyperinvariant subspaces does not apply, in terms that resemble a characterization of quasinilpotency. This motivates the following nomenclature.

Definition 1.4. Let \mathcal{H} be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. We say T is *s.o.t.-quasinilpotent* if

$$\text{s.o.t.-}\lim_{n \rightarrow \infty} ((T^*)^n T^n)^{1/2n} = 0.$$

Clearly, every quasinilpotent element is s.o.t.-quasinilpotent, and the hyperinvariant subspace problem for elements of II_1 -factors is reduced to the question for s.o.t.-quasinilpotent operators in II_1 -factors. Furthermore, the analogues of Questions 1.1 and 1.2 where “s.o.t.-quasinilpotent” replaces “quasinilpotent” are interesting, and we will answer positively the second of these.

Before we describe our main results, here some interesting examples related to Question 1.1.

Example 1.5. Let $\{x_1, x_2\}$ be free semicircular operators that generate the free group factor $L(\mathbb{F}_2)$. Then x_1 and x_2 are real parts of quasinilpotent operators in $L(\mathbb{F}_2)$. Indeed, $x_i/2$ is the real part of a copy of the quasinilpotent DT-operator in $L(\mathbb{F}_2)$, by results of [3].

Example 1.6. G. Tucci [8] found a family $(A_\alpha)_{0 < \alpha < 1}$ of quasinilpotent elements of the hyperfinite II_1 -factor R , each generating R as a von Neumann algebra. He showed for each α , $\text{Re}(A_\alpha)$ has the same moments as $\text{Im}(A_\alpha)$ and he found a combinatorial formula for them. He showed that each $\text{Re}(A_\alpha)$ generates a diffuse subalgebra of a Cartan masa in R , which is for some values of α all of the Cartan masa and for other values is a proper subalgebra of it.

Now we describe our main results. It is straightforward to see (the details can be found in Section 2) that if a diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ has zero trace, then the conjugate $B = U_n^* D U_n$ of this matrix by the $n \times n$ Fourier matrix $U_n = \frac{1}{\sqrt{n}}(\omega_n^{(j-1)(k-1)})_{1 \leq j, k \leq n}$, where $\omega_n = e^{2\pi i/n}$, is a Toeplitz matrix (meaning the (i, j) th entry depends only on $i - j$), has all zeros on the diagonal, and the upper

triangular part of it, which we will call T_λ , satisfies $T_\lambda + T_\lambda^* = B$. Here λ denotes the sequence $(\lambda_1, \dots, \lambda_n)$ and we have $\|D\| = \|\lambda\|_\infty := \max_j |\lambda_j|$. Note that T_λ is in fact an upper triangular Toeplitz matrix, and is nilpotent. A key issue is: how large is the norm of T_λ compared to the norm of D ?

The matrix T_λ is the image of B under the upper triangular truncation operator. The asymptotic behaviour of the norm of this upper triangular truncation operator on the $n \times n$ matrices was determined by Angelos, Cowen and Narayan in [1] to be $\frac{1}{\pi} \log(n) + O(1)$ as $n \rightarrow \infty$ (see Example 4.1 of [2] and [6] for earlier results). Our main result (Theorem 2.7) is that there is a constant K such that for every natural number n and every finite real sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ that sums to zero, there is a rearrangement $\tilde{\lambda}$ of λ such that $\|T_{\tilde{\lambda}}\| \leq K \|\lambda\|_\infty$. A value for the constant $K < 1.78$ (though not, to our knowledge, the best possible value) and the rearrangement $\tilde{\lambda}$ are found explicitly. The only requirement on the rearrangement is that the partial sums of the rearranged sequence do not exceed $\|\lambda\|_\infty$ in absolute value.

We observe that rearrangement is necessary by making the estimate (Proposition 2.8) that when $\lambda = (1, \dots, 1, -1, \dots, -1)$ of length $2n$ sums to zero, then we have $\|T_\lambda\| \geq \frac{1}{\pi} \log(2n) + C$ for a constant C , independent of n . In fact, the same asymptotic lower bound estimate, but for some different upper triangular Toeplitz matrices, was obtained by Angelos, Cowen and Narayan [1].

We also prove a slightly different rearrangement result of a similar nature (Proposition 2.10), for use in taking inductive limits.

In Section 3, we apply our main theorem to give some results in type I von Neumann algebras related to Question 1.2 and also draw some consequences in II_1 -factors.

In Section 4, we apply the related rearrangement result in an inductive limit to prove results about II_1 -factors. Finally, we ask a further specific question.

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2. UPPER TRIANGULAR TOEPLITZ MATRICES

For $n \in \mathbb{N}$ and $p \in \{0, 1, \dots, n\}$, let M_n denote the set of $n \times n$ matrices with complex values and let $\text{UT}_n^{(p)}$ denote the set of matrices $x \in M_n$ that have zero entries everywhere below the diagonal and on the first p diagonals on and above the main diagonal. That is, $x = (x_{ij})_{1 \leq i, j \leq n}$ belongs to $\text{UT}_n^{(p)}$ if and only if $x_{ij} = 0$ whenever $j < i + p$. So a matrix is strictly upper triangular if and only if it belongs to $\text{UT}_n^{(1)}$.

An $n \times n$ matrix $X = (x_{ij})_{1 \leq i, j \leq n} \in M_n$ is said to be a *Toeplitz matrix* if x_{ij} depends only on $i - j$. We let $\text{UTTM}_n^{(p)}$ be the set of all Toeplitz matrices that belong to $\text{UT}_n^{(p)}$. Every $T \in \text{UTTM}_n^{(1)}$ is of the form

$$T = \begin{pmatrix} 0 & t_1 & t_2 & \cdots & t_{n-1} \\ & 0 & t_1 & \ddots & \vdots \\ & & \ddots & \ddots & t_2 \\ & & & 0 & t_1 \\ & & & & 0 \end{pmatrix} \quad (2)$$

and is nilpotent. Moreover, $\text{UTTM}_n^{(0)}$ is a commutative algebra.

We now describe in more detail the matrices T_λ mentioned in the introduction. Let $\omega_n = e^{2\pi i/n}$. Recall that then $\sum_{j=0}^{n-1} \omega_n^{dj} = 0$ whenever d is an integer that is not divisible by n and, consequently, f_1, f_2, \dots, f_n is an orthonormal basis for \mathbb{C}^n , where

$$f_k = \frac{1}{\sqrt{n}} \sum_{j=1}^n \omega_n^{(k-1)(j-1)} e_j.$$

Let \mathcal{V}_n be the real vector space consisting of all real sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\sum_{j=1}^n \lambda_j = 0$.

For $\lambda \in \mathcal{V}_n$, consider the matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ with respect to the standard basis in \mathbb{C}^n , and let us write it as a matrix, B , with respect to the basis f_1, \dots, f_n . We have

$$\langle Df_k, f_l \rangle = \frac{1}{n} \left\langle \sum_{p=1}^n \lambda_p \omega_n^{(k-1)(p-1)} e_p, \sum_{q=1}^n \omega_n^{(l-1)(q-1)} e_q \right\rangle = \frac{1}{n} \sum_{p=1}^n \lambda_p \omega_n^{(k-l)(p-1)}. \quad (3)$$

Then the change-of-basis matrix whose columns are f_1, \dots, f_n is the $n \times n$ Fourier matrix, $U_n = \frac{1}{\sqrt{n}} (\omega_n^{(j-1)(k-1)})_{1 \leq j, k \leq n}$, and we have

$$B = U_n^* D U_n = \begin{pmatrix} 0 & t_1 & t_2 & \cdots & t_{n-1} \\ \overline{t_1} & 0 & t_1 & \ddots & \vdots \\ \overline{t_2} & \overline{t_1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 & t_1 \\ \overline{t_{n-1}} & \cdots & \cdots & \overline{t_1} & 0 \end{pmatrix} \quad (4)$$

is a Toeplitz matrix, where

$$t_d = t_d(\lambda) = \frac{1}{n} \sum_{p=1}^n \lambda_p \omega_n^{d(p-1)}. \quad (5)$$

Let $T_\lambda \in \text{UTTM}_n^{(1)}$ be the upper triangular part of B . By construction, $T_\lambda + T_\lambda^*$ is a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Thus, the map $\mathcal{V}_n \ni \lambda \mapsto T_\lambda \in \text{UTTM}_n^{(1)}$ is linear and injective.

Remark 2.1. From (5) we see

$$t_{n-d} = \overline{t_d}, \quad (1 \leq d \leq n-1). \quad (6)$$

Considering dimensions, we see that the map $\lambda \rightarrow T_\lambda$ is a linear isomorphism from \mathcal{V}_n onto the set of complex upper triangular Toeplitz matrices of the form (2) for which (6) holds.

For $\lambda \in \mathcal{V}_n$, let $\|\lambda\|_\infty = \max_j |\lambda_j|$. We regard such sequences as maps from $\{1, \dots, n\}$ to \mathbb{R} and, thus, for $\sigma \in S_n$, i.e., σ a permutation of $\{1, \dots, n\}$, $\lambda \circ \sigma \in \mathcal{V}_n$ denotes the sequence $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$. As described in the introduction, we will find

a constant K , independent of n , such that for every $\lambda \in \mathcal{V}_n$, there is $\sigma \in S_n$ such that

$$\|T_{\lambda \circ \sigma}\| \leq K \|\lambda\|_\infty = K \|B\|. \quad (7)$$

We will require some elementary lemmas. The next lemma is a simple observation about a known series expansion of the cotangent function.

Lemma 2.2. (i) For $x \in (0, 1)$,

$$\cot(\pi x) = \frac{1}{\pi x} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{2x}{k^2 - x^2}.$$

(ii) The function

$$f_1(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{2x}{k^2 - x^2}$$

increases on the set $[0, \frac{1}{2}]$ to the maximum value $\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{2}{\pi}$.

Lemma 2.3. Let $n \in \mathbb{N}$ and let

$$H_n = \frac{1}{n} \begin{pmatrix} 0 & \mathbf{i} & \mathbf{i} & \cdots & \mathbf{i} \\ -\mathbf{i} & 0 & \mathbf{i} & & \vdots \\ -\mathbf{i} & -\mathbf{i} & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{i} \\ -\mathbf{i} & \cdots & \cdots & -\mathbf{i} & 0 \end{pmatrix}$$

be the $n \times n$ self-adjoint matrix having zeros on the diagonal and all entries above the diagonal equal to $\mathbf{i} := \sqrt{-1}$. Then an orthonormal list of eigenvectors of H_n is $(v_k)_{k=0}^{n-1}$, and the associated eigenvalues are $(\mu_k)_{k=0}^{n-1}$, where if n is odd, then

$$\begin{aligned} v_k &= \frac{1}{\sqrt{n}} (1, -\omega_n^k, (-\omega_n^k)^2, \dots, (-\omega_n^k)^{n-1})^t, \\ \mu_k &= \frac{\mathbf{i}}{n} \sum_{j=1}^{n-1} (-\omega_n^k)^j = \frac{1}{n} \tan\left(\frac{\pi k}{n}\right) \end{aligned} \quad (8)$$

and we have $\mu_0 = 0$ and $\mu_k = -\mu_{n-k}$ if $1 \leq k \leq n-1$, while if n is even, then

$$\begin{aligned} v_k &= \frac{1}{\sqrt{n}} (1, \omega_n^k \omega_{2n}, (\omega_n^k \omega_{2n})^2, \dots, (\omega_n^k \omega_{2n})^{n-1})^t, \\ \mu_k &= \frac{\mathbf{i}}{n} \sum_{j=1}^{n-1} (\omega_n^k \omega_{2n})^j = -\frac{1}{n} \cot\left(\frac{\pi(2k+1)}{2n}\right) \end{aligned} \quad (9)$$

and we have $\mu_{n-1-k} = -\mu_k$ for all $0 \leq k \leq n-1$. Note that in all cases, we have $v_k = W_n^k v_0$ for all $0 \leq k \leq n-1$, where

$$W_n = \text{diag}(1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}).$$

Proof. One calculates $H_n v_k$ to be the vector whose p th entry, for $1 \leq p \leq n$, is

$$\begin{cases} \frac{i}{n\sqrt{n}} \left(-\sum_{j=0}^{p-2} (-\omega_n^k)^j + \sum_{j=p}^{n-1} (-\omega_n^k)^j \right), & n \text{ odd} \\ \frac{i}{n\sqrt{n}} \left(-\sum_{j=0}^{p-2} (\omega_n^k \omega_{2n})^j + \sum_{j=p}^{n-1} (\omega_n^k \omega_{2n})^j \right), & n \text{ even,} \end{cases} \quad (10)$$

where the sum $\sum_{j=0}^{p-2}$ is taken to be zero if $p = 1$, as is the sum $\sum_{j=p}^{n-1}$ if $p = n$. The quantity (10) equals $\frac{1}{\sqrt{n}}(-\omega_n^k)^{p-1}$ or, respectively, $\frac{1}{\sqrt{n}}(\omega_n^k \omega_{2n})^{p-1}$ times μ_k , for n odd and, respectively, even. This shows that v_k is an eigenvector with eigenvalue μ_k .

By standard properties of geometric progressions and trigonometry, we derive for n odd

$$\mu_k = \frac{i}{n} \cdot \frac{1 - \omega_n^k}{1 + \omega_n^k} = \frac{2}{n} \cdot \frac{\operatorname{Im}(\omega_n^k)}{|1 + \omega_n^k|^2} = \frac{1}{n} \tan \left(\frac{\pi k}{n} \right)$$

and for n even

$$\mu_k = \frac{i}{n} \cdot \frac{\omega_{2n}^{2k+1} + 1}{1 - \omega_{2n}^{2k+1}} = \frac{-2}{n} \cdot \frac{\operatorname{Im}(\omega_{2n}^{2k+1})}{|1 - \omega_{2n}^{2k+1}|^2} = -\frac{1}{n} \cot \left(\frac{\pi(2k+1)}{2n} \right).$$

All other assertions follow easily. \square

Remark 2.4. From Lemma 2.3 we get

$$H_n = \sum_{k=1}^n \mu_k W_n^k Q_n (W_n^*)^k,$$

where Q_n is the rank-one projection onto the span of the vector v_0 , and the n rank-one projections $W_n^k Q_n (W_n^*)^k$ for $1 \leq k \leq n$ are pairwise orthogonal. (Note: we let $\mu_n = \mu_0$ and $v_n = v_0$, while of course W_n^n is the identity matrix.)

The following facts follow directly from the formulas (8) and (9).

Remark 2.5. If n is odd, then the sequence $(\mu_k)_{k=1}^n$ is of the form

$$(a_1, a_2, \dots, a_{(n-1)/2}, -a_{(n-1)/2}, \dots, -a_2, -a_1, 0),$$

where

$$0 < a_1 < a_2 < \dots < a_{(n-1)/2} < \frac{2}{\pi},$$

while if n is even then $(\mu_k)_{k=0}^{n-1}$ is of the form

$$(-b_1, -b_2, \dots, -b_{n/2}, b_{n/2}, \dots, b_2, b_1),$$

where

$$\frac{2}{\pi} > b_1 > b_2 > \dots > b_{n/2} > 0.$$

The next lemma is just the well known scheme behind Dirichlet's test. We include the proof for convenience.

Lemma 2.6. *Let $n \in \mathbb{N}$ and suppose $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ where the sequence a_1, \dots, a_n monotone and $b_1 + \dots + b_n = 0$. Let*

$$M = \max_{1 \leq k \leq n} \left| \sum_{j=1}^k b_j \right|.$$

Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq M |a_n - a_1|.$$

Proof. Let $B_k = \sum_{j=1}^k b_j$, with $B_0 = 0$. By hypothesis, $B_n = 0$. Then

$$S := \sum_{j=1}^n a_j b_j = \sum_{j=1}^n a_j (B_j - B_{j-1}) = \sum_{j=1}^{n-1} B_j (a_j - a_{j+1}).$$

Using monotonicity of a_1, \dots, a_n , we get

$$|S| \leq \sum_{j=1}^{n-1} M |a_j - a_{j+1}| = M |a_1 - a_n|.$$

□

The next theorem shows that a constant $K < 1.78$ can be obtained in (7).

Theorem 2.7. *Let $n \in \mathbb{N}$ and let $\lambda \in \mathcal{V}_n$. Then there is a permutation $\sigma \in S_n$ such that*

$$\|T_{\lambda \circ \sigma}\| \leq K \|\lambda\|_\infty \quad (11)$$

with $K = \frac{1}{2} + \frac{4}{\pi}$.

Proof. Since λ is the eigenvalue sequence of $T_{\lambda \circ \sigma} + T_{\lambda \circ \sigma}^*$ for every σ , we have $\|\operatorname{Re} T_{\lambda \circ \sigma}\| = \frac{1}{2} \|\lambda\|_\infty$. Thus, it will suffice to find σ so that

$$\|\operatorname{i}(T_{\lambda \circ \sigma} - T_{\lambda \circ \sigma}^*)\| \leq \frac{8}{\pi} \|\lambda\|_\infty.$$

Using (5), we have

$$\begin{aligned} \operatorname{i}(T_\lambda - T_\lambda^*) &= \operatorname{i} \begin{pmatrix} 0 & t_1 & t_2 & \cdots & t_{n-1} \\ -\overline{t_1} & 0 & t_1 & \ddots & \vdots \\ -\overline{t_2} & -\overline{t_1} & 0 & \ddots & t_2 \\ \vdots & \ddots & \ddots & \ddots & t_1 \\ -\overline{t_{n-1}} & \cdots & -\overline{t_2} & -\overline{t_1} & 0 \end{pmatrix} \\ &= \frac{\operatorname{i}}{n} \sum_{p=1}^n \lambda_p \begin{pmatrix} 0 & \omega_n^{p-1} & \omega_n^{2(p-1)} & \cdots & \omega_n^{(n-1)(p-1)} \\ -\overline{\omega_n}^{p-1} & 0 & \omega_n^{p-1} & \ddots & \vdots \\ -\overline{\omega_n}^{2(p-1)} & -\overline{\omega_n}^{p-1} & 0 & \ddots & \omega_n^{2(p-1)} \\ \vdots & \ddots & \ddots & \ddots & \omega_n^{p-1} \\ -\overline{\omega_n}^{(n-1)(p-1)} & \cdots & -\overline{\omega_n}^{2(p-1)} & -\overline{\omega_n}^{p-1} & 0 \end{pmatrix} \\ &= \sum_{p=1}^n \lambda_p (W_n^*)^{p-1} H_n W_n^{p-1}. \end{aligned}$$

Using now Remark 2.4, we have

$$\mathrm{i}(T_\lambda - T_\lambda^*) = \sum_{p=1}^n \lambda_p \sum_{k=1}^n \mu_k W_n^{k-p+1} Q_n W_n^{p-k-1} = \sum_{l=1}^n \left(\sum_{k=1}^n \lambda_{k-l+1} \mu_k \right) W_n^l Q_n (W_n^*)^l,$$

where $k-l+1$ in the subscript of λ is taken modulo n in the range from 1 to n . Consequently,

$$\|T_\lambda - T_\lambda^*\| = \max_{1 \leq l \leq n} \left| \sum_{k=1}^n \lambda_{k-l+1} \mu_k \right| = \max_{1 \leq l \leq n} |(\lambda \circ \rho_n^{l-1}) \cdot \mu|,$$

where \cdot represents the usual scalar product, ρ_n is the full cycle permutation $\rho_n(j) = j-1 \pmod{n}$ and $\mu = (\mu_1, \dots, \mu_n)$. We seek a permutation σ making

$$\max_{1 \leq l \leq n} |(\lambda \circ \sigma \circ \rho_n^{l-1}) \cdot \mu| \leq \frac{8}{\pi} \|\lambda\|_\infty,$$

since the quantity on the left is $\|T_{\lambda \circ \sigma} - T_{\lambda \circ \sigma}^*\|$. Since we run through all rotations ρ_n^{l-1} , we may without loss of generality replace μ by $\mu \circ \rho_n^m$ for any m . From Remark 2.5, we see that some such $\mu \circ \rho_n^m$ is monotone, with largest element $< 2/\pi$ and smallest element $> -2/\pi$. Now we choose σ so that all partial sums of $\lambda \circ \sigma$ are of absolute value $\leq \|\lambda\|_\infty$. This implies that all partial sums of all rotations $\lambda \circ \sigma \circ \rho_n^{l-1}$ are of absolute value $\leq 2\|\lambda\|_\infty$. Now Lemma 2.6 implies

$$\|T_{\lambda \circ \sigma} - T_{\lambda \circ \sigma}^*\| \leq \frac{8}{\pi} \|\lambda\|_\infty.$$

□

The following result demonstrates that some rearrangement is required to get a bounded constant K in (7). Although this sort of calculation (to get a lower bound for the norm of the upper triangular projection) was also made in [1] for upper triangular Toeplitz matrices, these were not of the form T_λ for $\lambda \in \mathcal{V}_n$ (see Remark 2.1).

Proposition 2.8. *For $\lambda = (\underbrace{1, \dots, 1}_n, \underbrace{-1, \dots, -1}_n)$, we have $\|T_\lambda\| \geq \frac{1}{\pi} \log(n) - \frac{3}{2\pi}$.*

Proof. For the given λ ,

$$\begin{aligned} t_d &= \frac{1}{n} \sum_{k=1}^n \omega_{2n}^{d(k-1)} - \frac{1}{n} \sum_{k=n+1}^{2n} \omega_{2n}^{d(k-1)} = \frac{1}{n} \cdot \frac{1 - \omega_{2n}^{nd}}{1 - \omega_{2n}^d} \\ &= \frac{1}{n} \cdot \frac{1 - (-1)^d}{1 - \omega_{2n}^d}. \end{aligned}$$

Hence,

$$t_d = \begin{cases} 0 & \text{if } d \text{ is even} \\ \frac{1}{n} (1 + \mathrm{i} \cot(\frac{\pi d}{2n})) & \text{if } d \text{ is odd.} \end{cases} \quad (12)$$

We will estimate from below the quadratic form of T_λ on the vector $g = \frac{1}{\sqrt{2n}} \underbrace{(1, 1, \dots, 1)}_{2n}$.

$$\langle T_\lambda g, g \rangle = \left\langle \left(\sum_{k=1}^{2n-1} t_k, \sum_{k=1}^{2n-2} t_k, \dots, t_1 + t_2, t_1, 0 \right), g \right\rangle = \frac{1}{2n} \sum_{k=1}^{2n-1} (2n - k) t_k. \quad (13)$$

From (12) and (13), we obtain

$$\begin{aligned} \operatorname{Re} \langle T_\lambda g, g \rangle &= \frac{1}{2n^2} \sum_{\substack{1 \leq j \leq 2n-1 \\ j \text{ odd}}} (2n - j) = \frac{1}{2}, \\ \operatorname{Im} \langle T_\lambda g, g \rangle &= \frac{1}{2n^2} \sum_{\substack{1 \leq j \leq 2n-1 \\ j \text{ odd}}} (2n - j) \cot \left(\frac{\pi j}{2n} \right) \\ &= \frac{1}{2n^2} \left(\sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} + \sum_{\substack{n+1 \leq j \leq 2n-1 \\ j \text{ odd}}} \right) (2n - j) \cot \left(\frac{\pi j}{2n} \right) \\ &= \frac{1}{2n^2} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} (2n - 2j) \cot \left(\frac{\pi j}{2n} \right). \end{aligned}$$

Application of Lemma 2.2 implies

$$\begin{aligned} \operatorname{Im} \langle T_\lambda g, g \rangle &= \frac{1}{\pi n} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} \frac{2n - 2j}{j} - \frac{1}{2n^2} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} (2n - 2j) f_1 \left(\frac{j}{2n} \right) \\ &\geq \frac{1}{\pi n} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} \frac{2n - 2j}{j} - \frac{1}{\pi n^2} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} (2n - 2j) \\ &= \frac{2}{\pi} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} \frac{1}{j} - \frac{4}{\pi n} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} 1 + \frac{2}{\pi n^2} \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} j \end{aligned}$$

and the standard computations

$$\sum_{\substack{1 \leq j \leq r \\ j \text{ odd}}} \frac{1}{j} > \frac{1}{2} \log(r + 1), \quad \sum_{\substack{1 \leq j \leq r \\ j \text{ odd}}} j = \left\lfloor \frac{r+1}{2} \right\rfloor^2$$

(both for arbitrary $r \in \mathbb{N}$, where $\lfloor \cdot \rfloor$ is the floor function) yield

$$\operatorname{Im} \langle T_\lambda g, g \rangle > \frac{1}{\pi} \log(n) - \frac{3}{2\pi}.$$

□

We will now consider the conjugation with Fourier matrices with a view to taking inductive limits of matrix algebras. Let $(e_{ij}^{(n)})_{1 \leq i, j \leq n}$ be the standard system of matrix

units for M_n . Let $\Theta_n : M_n \rightarrow \text{UT}_n^{(1)}$ be the projection given by

$$\Theta_n(e_{ij}^{(n)}) = \begin{cases} e_{ij}^{(n)} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha_n : M_n \rightarrow M_n$ be the inner automorphism $\alpha_n(A) = U_n^* A U_n$, where U_n is the Fourier matrix as described above equation (4). Thus, for $\lambda = (\lambda_1, \dots, \lambda_n) \in D_n$ with $\sum_{j=1}^n \lambda_j = 0$, we have $T_\lambda = \Theta_n(\alpha_n(\lambda)) \in \text{UT}_n^{(1)}$.

Let $m, n \in \mathbb{N}$ and consider the inclusion $\gamma_{m,n} : M_m \rightarrow M_{mn}$ given by

$$\gamma = \gamma_{m,n} : e_{ij}^{(m)} \mapsto \sum_{k=1}^n e_{n(i-1)+k, n(j-1)+k}^{(mn)}.$$

Under the common identification of M_{mn} with $M_m \otimes M_n$, $\gamma_{m,n}(x)$ is identified with $x \otimes I_n$.

Let $\beta = \beta_{m,n} = \alpha_{mn}^{-1} \circ \gamma \circ \alpha_m : M_m \rightarrow M_{mn}$. While $\beta(e_{ij}^{(m)})$ is not very pretty to describe for general i and j , as seen in the following lemma, the restriction of β to the diagonal subalgebra D_m is rather nice; it is the flip of the usual tensor product embedding.

Lemma 2.9. *The map $\gamma_{m,n}$ sends Toeplitz matrices into Toeplitz matrices. For each $p \in \{0, 1, \dots, m\}$ we have*

$$\gamma_{m,n}(\text{UT}_m^{(p)}) \subseteq \text{UT}_{mn}^{(pn)}, \quad \gamma_{m,n}(\text{UTTM}_m^{(p)}) \subseteq \text{UTTM}_{mn}^{(pn)}. \quad (14)$$

Thus, we have the commuting diagram

$$\begin{array}{ccc} \text{UT}_m^{(1)} & \xrightarrow{\gamma} & \text{UT}_{mn}^{(1)} \\ \Theta_m \uparrow & & \uparrow \Theta_{mn} \\ M_m & \xrightarrow{\gamma} & M_{mn} \\ \alpha_m \uparrow & & \uparrow \alpha_{mn} \\ M_m & \xrightarrow{\beta} & M_{mn} \\ \uparrow & & \uparrow \\ D_m & \xrightarrow{\beta} & D_{mn} \end{array} \quad (15)$$

where the top and bottom row arrows indicate the restriction of γ to $\text{UT}_m^{(1)}$ and, respectively, β to D_m . Finally, for $e_{ii}^{(m)} \in D_m$, we have

$$\beta(e_{ii}^{(m)}) = \sum_{k=1}^n e_{m(k-1)+i, m(k-1)+i}^{(mn)}. \quad (16)$$

Proof. The inclusions (14) are easy to verify, and we need only show (16), which we do by verifying

$$\gamma \circ \alpha_m(e_{ii}^{(m)}) = \alpha_{mn} \left(\sum_{k=1}^n e_{m(k-1)+i, m(k-1)+i}^{(mn)} \right). \quad (17)$$

Using (3), we have

$$e_{ll}^{(m)} \xrightarrow{\alpha_m} \frac{1}{m} \sum_{1 \leq i, j \leq m} \omega_m^{(l-1)(j-i)} e_{ij}^{(m)} \xrightarrow{\gamma} \frac{1}{m} \sum_{\substack{1 \leq i, j \leq m \\ 1 \leq k \leq n}} \omega_m^{(l-1)(j-i)} e_{n(i-1)+k, n(j-1)+k}^{(mn)}, \quad (18)$$

while

$$\begin{aligned} \alpha_{mn} \left(\sum_{k=1}^n e_{m(k-1)+l, m(k-1)+l}^{(mn)} \right) &= \frac{1}{mn} \sum_{\substack{1 \leq k \leq n \\ 1 \leq a, b \leq mn}} \omega_{mn}^{(m(k-1)+l-1)(b-a)} e_{ab}^{(mn)} \\ &= \frac{1}{m} \sum_{1 \leq a, b \leq mn} \omega_{mn}^{(l-1)(b-a)} e_{ab}^{(mn)} \left(\frac{1}{n} \sum_{1 \leq k \leq n} \omega_n^{(k-1)(b-a)} \right). \end{aligned} \quad (19)$$

But

$$\frac{1}{n} \sum_{1 \leq k \leq n} \omega_n^{(k-1)(b-a)} = \begin{cases} 1, & \text{if } n \text{ divides } b-a \\ 0, & \text{if } n \text{ does not divide } b-a, \end{cases}$$

and n divides $b-a$ if and only if we have $a = n(i-1) + k$ and $b = n(j-1) + k$ for some $1 \leq i, j \leq m$ and some $1 \leq k \leq n$, so the nested summation (19) equals the right-most summation in (18), and (17) is verified. \square

For applications in the setting of inductive limits of maps as in the diagram (15), we will want a version of Theorem 2.7 but for elements of D_{mn} that are orthogonal to $\beta(D_m)$ and taking only reorderings of diagonal entries that fix $\beta(D_m)$. This is provided by the next result in the case $n = 2$.

Proposition 2.10. *Fix $m \in \{2, 3, \dots\}$ and let $\lambda_j^{(i)} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq 2$ satisfy $\lambda_1^{(i)} + \lambda_2^{(i)} = 0$ for all i . Thus,*

$$\lambda := (\lambda_1^{(1)}, \lambda_1^{(2)}, \dots, \lambda_1^{(m)}, \lambda_2^{(1)}, \lambda_2^{(2)}, \dots, \lambda_2^{(m)}) \in D_{2m} \ominus \beta_{m,2}(D_m).$$

Then there are permutations $\sigma_1, \dots, \sigma_m$ of $\{1, 2\}$ such that, considering the reordering

$$\kappa = (\lambda_{\sigma_1(1)}^{(1)}, \lambda_{\sigma_2(1)}^{(2)}, \dots, \lambda_{\sigma_m(1)}^{(m)}, \lambda_{\sigma_1(2)}^{(1)}, \lambda_{\sigma_2(2)}^{(2)}, \dots, \lambda_{\sigma_m(2)}^{(m)}),$$

of λ , we have $\|T_\kappa\| \leq C\|\lambda\|_\infty$, where $C = \frac{1}{2} + \frac{12}{\pi}$.

Proof. Proceeding as in the proof of Theorem 2.7 we have

$$\|T_\kappa\| \leq \frac{1}{2}\|\lambda\|_\infty + \frac{4}{\pi}R,$$

where

$$R = \max_{\substack{1 \leq k \leq 2m \\ 1 \leq p \leq 2m}} \left| \sum_{j=1}^k \kappa \circ \rho_{2m}^p(j) \right| \quad (20)$$

is the maximum absolute value of all partial sums of all rotations of κ , i.e., where ρ_{2m} is the full cycle permutation $k \mapsto k-1$ (modulo $2m$) of $\{1, \dots, 2m\}$. Since $\lambda_1^{(i)} = -\lambda_2^{(i)}$

for all i , we may choose σ_1 arbitrarily and then choose $\sigma_2, \dots, \sigma_m$ recursively so that the sign of $\lambda_{\sigma_k(1)}^{(k)}$ is the opposite of the sign of $\sum_{i=1}^{k-1} \lambda_{\sigma_i(1)}^{(i)}$. This ensures

$$\left| \sum_{i=1}^k \lambda_{\sigma_i(1)}^{(i)} \right| \leq \|\lambda\|_\infty$$

for all $k \in \{1, \dots, m\}$. This, in turn, implies

$$\left| \sum_{i=k}^l \lambda_{\sigma_i(j)}^{(i)} \right| \leq 2\|\lambda\|_\infty$$

for all $1 \leq k \leq l \leq m$ and all $j \in \{1, 2\}$. Together, these estimates yield $R \leq 3\|\lambda\|_\infty$. \square

3. APPLICATIONS USING FINITE TYPE I VON NEUMANN ALGEBRAS

This section is concerned with applications of Theorem 2.7 to constructing quasinilpotent and related elements in finite type I von Neumann algebras, and also constructions in II_1 factors that result from this. Throughout, K will denote the constant from Theorem 2.7.

Proposition 3.1. *Let $\mathcal{M}_j \subseteq \mathcal{B}(\mathcal{H}_j)$ be a von Neumann algebra ($j \in J$). Let $\mathcal{M} = \prod_{j \in J} \mathcal{M}_j$ be the direct product of von Neumann algebras, so that $\mathcal{M} \subseteq \mathcal{B}(\bigoplus_{j \in J} \mathcal{H}_j)$ in the canonical way. Let $x = (x_j)_{j \in J} \in \mathcal{M}$. Suppose each x_j is quasinilpotent.*

- (i) *Then x is s.o.t.-quasinilpotent.*
- (ii) *The element x is quasinilpotent if and only if*

$$\lim_{n \rightarrow \infty} \left(\sup_{j \in J} \|x_j^n\| \right)^{1/n} = 0. \quad (21)$$

- (iii) *If each x_j is nilpotent with $x_j^{n(j)} = 0$ for $n(j) \in \mathbb{N}$ and if*

$$\lim_{N \rightarrow \infty} (\sup \{\|x_j\| \mid j \in J, n(j) > N\}) = 0,$$

then x is quasinilpotent.

Proof. If $\xi = (\xi_j)_{j \in J} \in \bigoplus_{j \in J} \mathcal{H}_j$ with $\xi_j = 0$ for all $j \in J \setminus F$, where F is a finite subset of J , then

$$\|((x^*)^n x^n)^{1/2n} \xi\| \leq \left(\max_{j \in F} \|x_j^n\|^{1/n} \right) \|\xi\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This implies

$$\text{s.o.t.-} \lim_{n \rightarrow \infty} ((x^*)^n x^n)^{1/2n} = 0,$$

proving (i).

Assertion (ii) results from the formula for $\|x^n\|$, while the hypothesis of (iii) implies (21). \square

Lemma 3.2. *Let $n \in \mathbb{N}$ and let $\mathcal{M} = L^\infty(X, \nu) \otimes M_n(\mathbb{C})$ be a type I_n von Neumann algebra with separable predual and let $a = a^* \in \mathcal{M}$. Then a is the real part of a quasinilpotent element in \mathcal{M} if and only if the center-valued trace of a is zero. In this case, there is $z \in \mathcal{M}$ with $z^* + z = a$, $z^n = 0$ and $\|z\| \leq K\|a\|$.*

Proof. We identify \mathcal{M} with the bounded, ν -measurable functions $X \rightarrow M_n(\mathbb{C})$. Then the center-valued trace of $a \in \mathcal{M}$ is just the scalar valued function $\text{tr}_n(a(x))$. If $a = z + z^*$ for $z \in \mathcal{M}$ with z quasinilpotent, then for almost every $x \in X$, $z(x)$ will be nilpotent and its n th power must vanish. In particular, the matrix trace of $z(x)$ will vanish for almost every x ; consequently, the center-valued trace of a is zero.

If $a = a^* \in \mathcal{M}$, then by standard arguments we can choose a ν -measurable unitary-valued function $V_n : X \rightarrow M_n(\mathbb{C})$ so that $V_n(x)a(x)V_n(x)^*$ is diagonal for all $x \in X$. Let $\lambda(x)$ be the diagonal entries, i.e., $V_n(x)a(x)V_n(x)^* = \text{diag}(\lambda(x))$. If the center-valued trace of a is zero, then the sum of $\lambda(x)$ is zero (for almost every x) and since we may change $V_n(x)$ in a measurable way to re-order the diagonal elements as needed, using Theorem 2.7, we have $\|T_{\lambda(x)}\| \leq K\|a(x)\|$. Then $z(x) = V_n(x)^*T_{\lambda(x)}V_n(x)$ is the desired nilpotent element. \square

Combining Lemma 3.2 and Proposition 3.1, we obtain the following, which is a partial answer to Question 1.2.

Proposition 3.3. *Let \mathcal{M} be a finite type I von Neumann algebra with separable predual. We may write*

$$\mathcal{M} = \prod_{n \in J} L^\infty(X_n, \nu_n) \otimes M_n(\mathbb{C})$$

for some $J \subseteq \mathbb{N}$ and some nonzero finite measure ν_n . If $a = z + z^$ for $z \in \mathcal{M}$ an s.o.t.-quasinilpotent element, then the center-valued trace of a is zero.*

Conversely, suppose $a = a^ \in \mathcal{M}$ and that the center-valued trace of a is zero.*

- (i) *Then $a = z + z^*$ for an s.o.t.-quasinilpotent element $z \in \mathcal{M}$ with $\|z\| \leq K\|a\|$.*
- (ii) *If J is finite or if J is infinite but $a = a^* = (a_n)_{n \in J} \in \mathcal{M}$ with*

$$a_n \in L^\infty(X_n, \nu_n) \otimes M_n(\mathbb{C}) \quad \text{and} \quad \lim_{J \ni n \rightarrow \infty} \|a_n\| = 0,$$

then there is quasinilpotent element $z \in \mathcal{M}$ with $z + z^ = a$ and $\|z\| \leq K\|a\|$.*

It looks like further progress in answering Question 1.2 using these techniques involving upper triangular Toeplitz matrices could be made only with better understanding of the behavior of $\|T_\lambda^n\|^{1/n}$ for large n and long λ .

By embedding finite type I von Neumann algebras into II_1 -factors and using Proposition 3.3, one can obtain many examples of self-adjoint elements in II_1 -factors that are real parts of quasinilpotents. Recall that the distribution of a self-adjoint element of a II_1 -factor is the probability measure that is the trace composed with spectral measure.

Proposition 3.4. *Let R be the hyperfinite II_1 -factor and let $D \subset R$ be its Cartan (i.e., diagonal) maximal abelian self-adjoint subalgebra. Suppose a compactly*

supported Borel probability measure ν on \mathbb{R} is of the form

$$\nu = \sum_{n \in J} \frac{1}{n} \int (\delta_{f_{n,1}(t)} + \delta_{f_{n,2}(t)} + \cdots + \delta_{f_{n,n}(t)}) d\nu_n(t), \quad (22)$$

where $J \subseteq \mathbb{N}$ or $J = \mathbb{N}$, where each ν_n is a nonzero positive measure on a standard Borel space X with $\sum_{j \in J} \nu_j(X) = 1$ and where $f_{n,1}, \dots, f_{n,n}$ are real-valued measurable functions on X such that for ν_n -almost every x we have $f_{n,1}(x) + \cdots + f_{n,n}(x) = 0$.

- (i) Then there is an s.o.t.-quasinilpotent element $z \in R$ such that $a := z + z^* \in D$, $\|z\| \leq K\|a\|$ and the distribution μ_a is equal to ν .
- (ii) If J is finite, then the element z can be chosen to be nilpotent.
- (iii) Suppose J is infinite and let

$$M_n = \max\{\|f_{n,1}\|_\infty, \dots, \|f_{n,n}\|_\infty\},$$

where the norms are in $L^\infty(\nu_n)$. If $\lim_{j \rightarrow \infty} M_j = 0$, then the element z can be chosen to be quasinilpotent.

Proof. We can realize D as a copy of

$$\prod_{n \in J} L^\infty(\nu_n)^{\oplus n}$$

in R , and using partial isometries from R we can find a type I subalgebra \mathcal{M} with $D \subseteq \mathcal{M} \subset R$ of the form

$$\mathcal{M} = \prod_{n \in J} L^\infty(\nu_n) \otimes M_n(\mathbb{C}),$$

where identifying each $L^\infty(\nu_n) \otimes M_n(\mathbb{C})$ with the $M_n(\mathbb{C})$ -valued ν_n -measurable functions, D is identified with the product of the sets of functions taking values in the diagonal matrices. The element $a = (\text{diag}(f_{n,1}(\cdot), \dots, f_{n,n}(\cdot)))_{n \in J}$ belongs to D , has center-valued trace in \mathcal{M} equal to zero and has distribution ν . Now we apply Proposition 3.3, to find $z \in \mathcal{M}$ having the desired properties. \square

The question of whether every self-adjoint element of $a \in D$ having distribution ν as in the above proposition is the real part of a quasinilpotent remains unanswered in general, though it is not hard to show that if the essential ranges of the functions $(f_{n,i})_{n \in J, 1 \leq i \leq n}$ are pairwise disjoint, then the answer is yes, by the construction used above. The similar question for arbitrary self-adjoint elements of R is even less clear. However, in the ultrapower of the hyperfinite II_1 -factor, the answer is yes.

Proposition 3.5. *Let R^ω be an ultrapower of the hyperfinite II_1 factor, for ω a non-principle ultrafilter on \mathbb{N} . Let $a = a^* \in R^\omega$ and suppose the distribution of a is of the form ν as in (22).*

- (i) *Then there is an s.o.t.-quasinilpotent element $z \in R^\omega$ such that $a = z + z^*$ and $\|z\| \leq K\|a\|$.*
- (ii) *If J is finite, then the element z can be chosen to be nilpotent.*
- (iii) *Suppose J is infinite and let*

$$M_n = \max\{\|f_{n,1}\|_\infty, \dots, \|f_{n,n}\|_\infty\},$$

where the norms are in $L^\infty(\nu_n)$. If $\lim_{j \rightarrow \infty} M_j = 0$, then the element z can be chosen to be quasinilpotent.

Proof. Since $R \subseteq R^\omega$ as a unital W^* -subalgebra, using Proposition 3.4, there is $b = b^* \in R^\omega$ whose distribution is ν and with s.o.t.-quasinilpotent $y \in R^\omega$ such that $b = y + y^*$ and $\|y\| \leq K\|b\|$, and according with the additional stipulations of (ii) and (iii) in the case that the corresponding hypotheses are satisfied. Since all self-adjoint elements in R^ω having given distribution are unitarily equivalent, we find z as a unitary conjugate of y . \square

For a purely spectral condition that is sufficient for a self-adjoint to be the real part of a quasinilpotent, valid in all II_1 -factors, we turn to discrete measures.

Proposition 3.6. *Let \mathcal{M} be a II_1 -factor with trace τ and let $a = a^* \in \mathcal{M}$ with $\tau(a) = 0$. Suppose that the distribution μ_a of a is a discrete measure that can be written*

$$\mu_a = \sum_{i \in I} s_i \left(\frac{1}{n(i)} \sum_{k=1}^{n(i)} \delta_{t(i,k)} \right),$$

where for all $i \in I$, $s_i > 0$, $n(i) \in \mathbb{N}$, $t(i, k) \in \mathbb{R}$, $\sum_{k=1}^{n(i)} t(i, k) = 0$ and where δ_t denotes the Dirac measure at t and $\sum_{i \in I} s_i = 1$.

- (i) Then there is an s.o.t.-quasinilpotent element $z \in \mathcal{M}$ such that $a = z + z^*$ and $\|z\| \leq K\|a\|$.
- (ii) If $\sup_{i \in I} n(i) < \infty$, then the element z can be chosen to be nilpotent.
- (iii) Let $M_i = \max\{|t(i, 1)|, \dots, |t(i, n(i))|\}$. If

$$\lim_{N \rightarrow \infty} (\sup\{M_i \mid i \in I, n(i) > N\}) = 0,$$

then the element z can be chosen to be quasinilpotent.

Proof. By Proposition 3.4, there is a quasinilpotent element $y \in R$ such that the distribution of $y + y^*$ equals μ_a and $\|y\| \leq K\|a\|$. There is a copy of R embedded as a unital W^* -subalgebra of \mathcal{M} . Since the spectral measure of a is discrete, all self-adjoint elements in \mathcal{M} having this spectral measure are unitarily equivalent in \mathcal{M} . Thus, a unitary conjugate of y is the desired element z . \square

4. APPLICATIONS USING INDUCTIVE LIMITS

In this section we will apply Proposition 2.10 in the setting of inductive limits of maps like the ones in (15) of Lemma 2.9, to conclude that some self-adjoint elements of the Cartan masa in the hyperfinite II_1 -factor R whose distributions are of a certain form, are the real parts of quasinilpotent elements in R .

We will use the following easy result to construct quasinilpotent elements.

Proposition 4.1. *Let z_1, z_2, \dots be pairwise commuting quasinilpotent elements in a Banach algebra B and suppose $\sum_{j=1}^{\infty} \|z_j\| < \infty$. Let $z = \sum_{j=1}^{\infty} z_j$. Then z is quasinilpotent.*

Proof. Without loss of generality we may take B unital. Let A be the unital Banach subalgebra generated by z_1, z_2, \dots . Then A is an abelian algebra. We need only to show that the spectrum of z relative to A is $\{0\}$ since it is equivalent to $\lim_{n \rightarrow \infty} \|z^n\|^{1/n} = 0$. Using the Gelfand transform,

$$\sigma_A(z) = \{\varphi(z) : \varphi \text{ is a multiplicative linear functional of } A\}.$$

Since z_n is quasinilpotent, we have $\varphi(z_n) = 0$ for every multiplicative linear functional φ on A . Since multiplicative linear functionals are automatically bounded, we have

$$\varphi(z) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \varphi(z_k) = 0,$$

which proves the lemma. \square

Consider the Cartan masa (maximal abelian self-adjoint subalgebra) D of the hyperfinite II_1 -factor R , realized as the inductive limit of the trace-preserving maps shown below,

$$\begin{array}{ccccccc} M_{n_1} & \xrightarrow{\beta^{(1)}} & M_{n_1 n_2} & \xrightarrow{\beta^{(2)}} & M_{n_1 n_2 n_3} & \xrightarrow{\beta^{(3)}} & \dots M_{n_1 n_2 \dots n_j} \xrightarrow{\beta^{(j)}} \dots R \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ D_{n_1} & \xrightarrow{\beta^{(1)}} & D_{n_1 n_2} & \xrightarrow{\beta^{(2)}} & D_{n_1 n_2 n_3} & \xrightarrow{\beta^{(3)}} & \dots D_{n_1 n_2 \dots n_j} \xrightarrow{\beta^{(j)}} \dots D, \end{array} \quad (23)$$

where $n_1, n_2, \dots \in \{2, 3, \dots\}$ and $\beta^{(j)}$ is the map $\beta_{n_1 n_2 \dots n_j, n_1 n_2 \dots n_j n_{j+1}}$ defined above Lemma 2.9, and whose restriction to the diagonal subalgebra $D_{n_1 n_2 \dots n_j}$ is as described in Lemma 2.9.

Lemma 4.2. *Suppose $a_j = a_j^* \in D_{n_1 \dots n_j}$ are such that $\tau(a_j) = 0$ for all j . Let $T_{a_j} = \Theta_{n_1 \dots n_j} \circ \alpha^{(j)}(a_j) \in \text{UTM}_{n_1 \dots n_j}^{(1)}$ and suppose $\sum_{j=1}^{\infty} \|T_{a_j}\| < \infty$. Then the series $a := \sum_{j=1}^{\infty} a_j \in D$ converges in norm and there is a quasinilpotent operator $z \in R$ such that $z^* + z = a$ and $\|z\| \leq \sum_{j=1}^{\infty} \|T_{a_j}\|$.*

Proof. Using Lemma 2.9, we have the big commuting diagram

$$\begin{array}{ccccccc} \text{UT}_{n_1} & \xrightarrow{\gamma^{(1)}} & \text{UT}_{n_1 n_2} & \xrightarrow{\gamma^{(2)}} & \text{UT}_{n_1 n_2 n_3} & \xrightarrow{\gamma^{(3)}} & \dots \text{UT}_{n_1 n_2 \dots n_j} \xrightarrow{\gamma^{(j)}} \dots \subset R \\ \uparrow \Theta^{(1)} & & \uparrow \Theta^{(2)} & & \uparrow \Theta^{(3)} & & \uparrow \Theta^{(j)} \\ M_{n_1} & \xrightarrow{\gamma^{(1)}} & M_{n_1 n_2} & \xrightarrow{\gamma^{(2)}} & M_{n_1 n_2 n_3} & \xrightarrow{\gamma^{(3)}} & \dots M_{n_1 n_2 \dots n_j} \xrightarrow{\gamma^{(j)}} \dots R \\ \uparrow \alpha^{(1)} & & \uparrow \alpha^{(2)} & & \uparrow \alpha^{(3)} & & \uparrow \alpha^{(j)} \\ M_{n_1} & \xrightarrow{\beta^{(1)}} & M_{n_1 n_2} & \xrightarrow{\beta^{(2)}} & M_{n_1 n_2 n_3} & \xrightarrow{\beta^{(3)}} & \dots M_{n_1 n_2 \dots n_j} \xrightarrow{\beta^{(j)}} \dots R \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ D_{n_1} & \xrightarrow{\beta^{(1)}} & D_{n_1 n_2} & \xrightarrow{\beta^{(2)}} & D_{n_1 n_2 n_3} & \xrightarrow{\beta^{(3)}} & \dots D_{n_1 n_2 \dots n_j} \xrightarrow{\beta^{(j)}} \dots D, \end{array} \quad (24)$$

where $\gamma^{(j)} = \gamma_{n_1 n_2 \dots n_j, n_1 n_2 \dots n_j n_{j+1}}$ is the usual inclusion of tensor products, where $\alpha^{(j)} = \alpha_{n_1 n_2 \dots n_j}$ is the automorphism implemented by conjugation with the Fourier

matrix and their inductive limit α is the resulting isomorphism between copies of the hyperfinite II_1 -factor, and where $\Theta^{(j)} = \Theta_{n_1 n_2 \dots n_j}$ is the upper triangular projection.

Since upper triangular Toeplitz matrices commute with each other, and taking into account the observation (14) of Lemma 2.9, by Proposition 4.1 the series $\tilde{z} := \sum_{j=1}^{\infty} T_{a_j}$ converges in norm to a quasinilpotent operator in R . By construction, we have $T_{a_j} + T_{a_j}^* = \alpha^{(j)}(a_j)$, so the series $a = \sum_{j=1}^{\infty} a_j$ converges in norm, and $a = z^* + z$, where $z = \alpha^{-1}(\tilde{z})$. \square

Lemma 4.3. *Suppose D is the Cartan masa of the hyperfinite II_1 -factor R and $a = a^* \in D$ has trace zero. Let $n_1 \in \mathbb{N}$, $n_2 = n_3 = \dots = 2$, and suppose there exists an increasing family*

$$D^{(1)} \subseteq D^{(2)} \subseteq D^{(3)} \subseteq \dots$$

of abelian, unital $$ -subalgebras of D whose union is weakly dense in D , and where each $D^{(j)}$ has dimension $n_1 n_2 \dots n_j$ and has minimal projections equally weighted by the trace. Letting $E_j : D \rightarrow D^{(j)}$ denote the trace-preserving conditional expectation, (with E_0 being simply the trace), suppose*

$$S := \sum_{j=1}^{\infty} \|E_j(a) - E_{j-1}(a)\| < \infty. \quad (25)$$

Then there is an automorphism σ of D and quasinilpotent element $z \in R$ so that $z^ + z = \sigma(a)$ and $\|z\| \leq CS$, where the constant C is from Proposition 2.10.*

Proof. We may write D as an inductive limit as in the bottom row of (23), where we now think of $D^{(j)}$ as the set of diagonal matrices in M_{2^j} and the inclusion $D^{(j)} \subseteq D^{(j+1)}$ given by the map β as in (16), with $m = n_1 2^{j-1}$ and $n = 2$. Then using that D is a Cartan masa in R , the inclusion $D \hookrightarrow R$ may be written as an inductive limit as in (23). Let $a_j = E_j(a) - E_{j-1}(a)$. Note that $E_0(a) = 0$ and we have $a = \sum_{j=1}^{\infty} a_j$, with the estimate (25) ensuring convergence in norm. By Proposition 2.10, for each j there is a trace-preserving automorphism σ_j of $D^{(j)}$ fixing each element of $D^{(j-1)}$ (if $j \geq 2$) and such that

$$\|T_{\sigma_j(a_j)}\| \leq C\|a_j\|.$$

The inductive limit of these automorphisms σ_j is an automorphism σ of D , and we have

$$\sigma(a) = \sum_{j=1}^{\infty} \sigma_j(a_j).$$

Now by Lemma 4.2, there is a quasinilpotent element $z \in R$ such that $z + z^* = \sigma(a)$ and $\|z\| \leq CS$. \square

We now provide examples of the elements a satisfying hypotheses of Lemma 4.3.

Proposition 4.4. *Let D be the Cartan masa of the hyperfinite II_1 -factor R , and let $a = a^*$ be an element in D with $\tau(a) = 0$. Suppose, in addition, that the distribution μ_a of a satisfies:*

(i) μ_a has at most a finite number of atoms, each of rational weight,

- (ii) the nonatomic part of μ_a is either zero or has support equal to the union of finitely many pairwise disjoint closed intervals I_j ,
- (iii) $\mu(I_j)$ is rational for each j ,
- (iv) the restriction of the nonatomic part of μ_a to each of the intervals I_j is Lebesgue absolutely continuous and has Radon–Nikodym derivative with respect to Lebesgue measure that is bounded below on I_j by some $\delta > 0$.

Then the element a satisfies the hypothesis of Lemma 4.3 and, consequently, $\sigma(a)$ is the real part of a quasinilpotent operator in R , for some automorphism σ of D .

Proof. Let n_1 be an integer large enough so that (a) n_1 times the weight of every atom of μ_a is an integer and (b) n_1 times each $\mu_a(I_j)$ is an integer. Then we may choose an n_1 dimensional subalgebra $D^{(1)}$ of D with minimal projections p_1, \dots, p_{n_1} that are equally weighted by the trace, and such that each $p_j a$ is either a scalar multiple of p_j or an element whose distribution is Lebesgue absolutely continuous on an interval with Radon–Nikodym derivative that is bounded below by δ on its support.

Now it suffices to consider a single element $b \in D$ whose distribution μ_b is Lebesgue absolutely continuous, is supported on a closed interval $[c, d]$, with $c < d$, and having Radon–Nikodym derivative with respect to Lebesgue measure that is bounded below by $\delta > 0$. It will suffice to find an increasing chain of subalgebras $D^{(j)}$ of dimension 2^j and with all minimal projections having trace 2^j , such that $\sum_{j=1}^{\infty} \|b - E_j(b)\| < \infty$, where E_j is the conditional expectation onto $D^{(j)}$. This is easily done. Indeed we have the partition $c = c_0^{(j)} < c_1^{(j)} < \dots < c_{2^j}^{(j)} = d$ of $[c, d]$ so that $\mu_b([c_{k-1}^{(j)}, c_k^{(j)}]) = 2^{-j}$ for all k . As the Radon–Nikodym derivative is bounded below by δ , we have $c_k^{(j)} - c_{k-1}^{(j)} < 2^{-j}/\delta$ for all k . Then letting $D^{(j)}$ be the subalgebra of D spanned by the spectral projections of b corresponding to the intervals $[c_{k-1}^{(j)}, c_k^{(j)}]$, we have $\|b - E_j(b)\| \leq 2^{-j}/\delta$ and $D^{(j)} \subseteq D^{(j+1)}$. \square

The techniques we have employed suggest the following question:

Question 4.5. If a is a self-adjoint element in the UHF algebra M_{2^∞} whose trace is zero, is a the real part of a quasi-nilpotent operator?

However, the key point for the previous proposition was to arrange that the series in (25) be summable. We do not see how to make this so for an arbitrary element of the diagonal of the UHF algebra M_{2^∞} embedded in R . The following example illustrates the difficulty.

Example 4.6. Let the Cartan masa D be identified with $L^\infty[-\frac{1}{2}, \frac{1}{2}]$ with the trace given by Lebesgue measure. Let $a \in D$ be the increasing function whose distribution μ_a is

$$\mu_a = \sum_{n=2}^{\infty} \frac{1}{2^n} (\delta_{-\frac{1}{n}} + \delta_{\frac{1}{n}}).$$

Thus, we have

$$a(t) = \begin{cases} -\frac{1}{n}, & -2^{-(n-1)} < t < -2^{-n}, \quad n \geq 2 \\ \frac{1}{n}, & 2^{-n} < t < 2^{-(n-1)}, \quad n \geq 2. \end{cases}$$

Let $D^{(n)}$ be the subalgebra of D that is spanned by the characteristic functions of the intervals

$$\left(-\frac{1}{2} + \frac{k-1}{2^n}, -\frac{1}{2} + \frac{k}{2^n}\right), \quad (1 \leq k \leq 2^n)$$

and let E_n denote the conditional expectation of D onto $D^{(n)}$. Let $s_N = \sum_{n=N+1}^{\infty} \frac{1}{n2^n}$. Then we have

$$E_N(a)(t) = \begin{cases} -\frac{1}{n}, & -2^{-(n-1)} < t < -2^{-n}, \quad 2 \leq n \leq N \\ -s_N, & -2^{-N} < t < 0 \\ s_N, & 0 < t < 2^{-N} \\ \frac{1}{n}, & 2^{-n} < t < 2^{-(n-1)}, \quad 2 \leq n \leq N. \end{cases}$$

and from this we compute $\|E_{N+1}(a) - E_N(a)\| = \max(s_N - s_{N+1}, |s_N - \frac{1}{N+1}|)$ for each $N \geq 1$. Since $0 < s_N < 2^{-N-1}$, we have

$$\sum_{N=1}^{\infty} \|E_{N+1}(a) - E_N(a)\| = \infty. \quad (26)$$

While the above example does not prove that no choice of subalgebras $D^{(n)}$ can be made which renders finite the corresponding sum (26), we do not see a choice that would do so.

The next proposition employs the usual techniques to give more examples in ultrapower II_1 -factors.

Proposition 4.7. *Let R^ω be an ultrapower of the hyperfinite II_1 factor, for ω a non-principle ultrafilter on \mathbb{N} . Let $a = a^* \in R^\omega$ have trace zero and suppose the distribution μ_a of a satisfies the hypotheses of Proposition 4.4. Then there is a quasinilpotent element $z \in R^\omega$ such that $a = z + z^*$.*

Proof. By Proposition 4.4, there is a quasinilpotent element $\tilde{z} \in R$ such that the distribution of $\tilde{a} := \tilde{z} + \tilde{z}^*$ equals μ_a . Thus, the element b of R^ω which is the class of the sequence of \tilde{a} repeated infinitely often is (a) equal to $y + y^*$ for a quasinilpotent element y of R^ω and (b) has distribution equal to μ_a . Since all the self-adjoint elements in R^ω having a given distribution are unitarily equivalent, we find the desired element z as a unitary conjugate of y . \square

Finally, here is a specific question

Question 4.8. Let p be a projection in the hyperfinite II_1 -factor or, for that matter, in any specific II_1 -factor, whose trace $\tau(p)$ is irrational. Is $p - \tau(p)1$ the real part of a quasinilpotent element of the II_1 -factor?

Of course, with $\tau(p)$ rational, the element $p - \tau(p)1$ is the real part of a nilpotent in an embedded matrix algebra. However, with $\tau(p)$ irrational, the techniques used in this paper do not apply to the element $p - \tau(p)1$, as it does not have the same distribution as any element with vanishing center-valued trace in a finite type I von Neumann algebra, nor does it fall under the rubric of results in this section.

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